Solution of Fokker-Planck equation for a broad class of drift and diffusion coefficients

Kwok Sau Fa

Departamento de Física, Universidade Estadual de Maringá,

Av. Colombo 5790, 87020-900, Maringá-PR,

Brazil, Tel.: 55 44 32614330, Fax: 55 44 32634623[*]

Abstract

We consider a Langevin equation with variable drift and diffusion coefficients separable in time and space and its corresponding Fokker-Planck equation in the Stratonovich approach. From this Fokker-Planck equation we obtain a class of exact solutions with the same spatial drift and diffusion coefficients. Furthermore, we analyze some details of this system by using the spatial diffusion coefficient $D(x) = \sqrt{D}\,|x|^{-\frac{\theta}{2}}$.

PACS numbers: 05.40.-a, 05.60.-k, 02.50.-r

Keywords: Langevin equation; Fokker-Planck equation; multiplicative noise; anomalous diffusion

[*]Electronic address: kwok@dfi.uem.br

In these last decades, anomalous diffusion properties have been extensively investigated by several approaches in order to model different kinds of probability distributions such as long-range spatial or temporal correlations [1, 2]. These approaches have been used to describe numerous systems in several contexts such as physics, hydrology, chemistry and biology. The diffusion process is classified according to the mean square displacement (MSD)

$$\left\langle x^2(t) \right\rangle \sim t^{\alpha} \ . \tag{1}$$

In the case of normal diffusion, the MSD grows linearly with time ($\alpha = 1$). For $0 < \alpha < 1$ the process is called subdiffusive, and for $\alpha > 1$ the process is called superdiffusive. The well-established property of the normal diffusion described by the Gaussian distribution can be obtained by the usual Fokker-Planck equation with a constant diffusion coefficient (without the drift term) [3, 4] or by an integro-differential diffusion equation with the exponential function for the waiting time probability distribution [5]. Anomalous diffusion regimes can also be obtained by the usual Fokker-Planck equation, however, they arise from variable diffusion coefficient which depends on time and/or space. On the other hand, in the view of Langevin approach it is associated with a multiplicative noise term. In other approaches such as the generalized Fokker-Planck equation (nonlinear) and fractional equations, they can describe anomalous diffusion regimes with a constant diffusion coefficient.

The Langevin equation is a very important tool for describing systems out of equilibrium [3, 4]. Moreover, this equation has been extensively investigated; many properties and analytical solutions of it have also been revealed. In this work, we present solutions of a class of the Langevin equation with the deterministic drift and multiplicative noise terms in time and space. To do so, we obtain the corresponding Fokker-Planck equation in the Stratonovich definition, and then we obtain its solutions for the probability distribution function (PDF).

Langevin equation and its corresponding Fokker-Planck equation. We consider the following Langevin equation in one-dimensional space with a multiplicative noise term:

$$\dot{\xi} = h(\xi, t) + g(\xi, t)\Gamma(t) \ , \tag{2}$$

where ξ is a stochastic variable and $\Gamma(t)$ is the Langevin force. We assume that the averages $\langle \Gamma(t) \rangle = 0$ and $\left\langle \Gamma(t)\Gamma(\bar{t}) \right\rangle = 2\delta(t - \bar{t})$ [3]. $h(\xi, t)$ is the deterministic drift. Physically, the additive noise ($g(\xi, t)$ constant) may represent the heat bath acting on the particle of the

system, and the multiplicative noise term, for variable $g(\xi,t)$, may represent a fluctuating barrier. For $g = \sqrt{D}$ and $h(\xi,t) = 0$, Eq. (2) describes the Wiener process and the corresponding probability distribution is described by a Gaussian function. In the case of $g(\xi,t)$, some specific functions have been employed to study, for instance, turbulent flows ($g(x,t) \sim |x|^a t^b$) [6–8]. By applying the Stratonovich approach in a one-dimensional space [3], we obtain the following dynamic equation for the PDF:

$$\frac{\partial W(x,t)}{\partial t} = -\frac{\partial}{\partial x}\left[D_1(x,t)W(x,t)\right] + \frac{\partial^2}{\partial x^2}\left[D_2(x,t)W(x,t)\right] \ , \tag{3}$$

where $D_1(x,t)$ and $D_2(x,t)$ are the drift and diffusion coefficients given by

$$D_1(x,t) = h(x,t) + \frac{\partial g(x,t)}{\partial x} g(x,t) \tag{4}$$

and

$$D_2(x,t) = g^2(x,t). \tag{5}$$

We note that Eq. (3) has a spurious drift due to the Stratonovich definition. Moreover, Eq. (3) can be written as

$$\frac{\partial W(x,t)}{\partial t} = -\frac{\partial}{\partial x}\left[h(x,t)W(x,t)\right] + \frac{\partial}{\partial x}\left[g(x,t)\frac{\partial g(x,t)W(x,t)}{\partial x}\right] \ . \tag{6}$$

For the case of $h(x,t) = 0$ and $g(x,t) = T(t)D(x)$ the system has been considered in Ref. [9]; the solution for $W(x,t)$ is given by

$$W(x,t) = B(t)\,\frac{\exp\left[-\frac{\overline{x}(x)^2}{4\overline{t}(t)}\right]}{D(x)\sqrt{\overline{t}\,(t)}} \ , \tag{7}$$

where

$$\frac{d\overline{t}}{dt} = T^2(t) \ , \tag{8}$$

$$\frac{d\overline{x}}{dx} = \frac{1}{D(x)} \tag{9}$$

and $B(t)$ is a normalization factor. Eq. (7) can describe interesting properties such as non-Gaussian distribution, combination of behaviors like Gaussian (for small distance) and exponential (for large distance), and combination of behaviors like Gaussian (for small distance) and power law decay for long distance. Further, it can describe many bimodal distributions for different forms of $g(x,t)$. For instance, let us consider $D(x) = \sqrt{D}\,|x|^{-\frac{\theta}{2}}$,

then the probability distribution and MSD are given by

$$W(x,\overline{t}) = |x|^{\frac{\theta}{2}} \frac{\exp\left[-\frac{|x|^{2+\theta}}{D(2+\theta)^2\overline{t}}\right]}{\sqrt{4\pi D\overline{t}}} \tag{10}$$

and

$$\left\langle x^2(t)\right\rangle = \frac{\left[D^2\left(2+\theta\right)^4\right]^{\frac{1}{2+\theta}}\Gamma\left(\frac{6+\theta}{2(2+\theta)}\right)\overline{t}^{\frac{2}{2+\theta}}(t)}{\sqrt{\pi}}, \ \theta > -2\ . \tag{11}$$

Moreover, the PDF can also be obtained for $\theta = -2$; in this case the PDF gives a log-normal distribution. One can see that the multiplicative noise term in space $D(x) = \sqrt{D}\,|x|^{-\frac{\theta}{2}}$ produces non-Gaussian shapes for the PDF (10); it presents a Gaussian shape only for $\theta = 0$. It can also reproduce the asymptotic behavior of the random-walk model and time fractional dynamic equation for $\overline{t} = t^{\beta(2+\theta)/2}$ [9], where $0 < \beta < 1$. Now we want to show two interesting processes which can be obtained from Eqs. (10) and (11). To do so, we take $\theta > -2$. The first one we consider a simple expression for $T(t)$ given by

$$T(t) = \frac{\sqrt{q}}{\sqrt{t}}, \tag{12}$$

for $t \gg 1$. From Eq. (8) yields

$$\overline{t}\,(t) = q\ln t\ . \tag{13}$$

Eqs. (11) and (13) describe the ultraslow diffusion processes. This kind of diffusion has been found, for instance, in aperiodic environments [10].

The second one we consider the following $T(t)$:

$$T(t) = \frac{\sqrt{\alpha t^{\alpha-1}}\sqrt{\sum_{j=0}^{n} c_j\lambda_j e^{-\lambda_j t^\alpha}}}{\sum_{i=0}^{n} c_i e^{-\lambda_i t^\alpha}}\quad , \tag{14}$$

where c_j, λ_j and α are constants. Using the function (14) one can obtain anomalous diffusion processes with logarithmic oscillations. We note that the time behavior with logarithmic oscillation is ubiquitous; examples have been observed, for instance, in epidemic spreading in fractal media [11], financial stock market [12] and diffusion-limited aggregates [13]. In Fig. 1 we show the function $T(t)$ (14) for $\lambda_i = a^i$, $c_i = (a/b)^i$, $a = 1/15$ and $b = 0.3$; for these values the curves present logarithmic oscillations with different values of n and α. From Eq. (8) we obtain

$$\overline{t}\,(t) = \frac{1}{\sum_{i=0}^{n} c_i e^{-\lambda_i t^\alpha}}\ . \tag{15}$$

Moreover, the PDF (10) presents unimodal states for $-2 < \theta \leq 0$ and bimodal states for $\theta > 0$ (see in Fig. 2) with pronounced cusps. The numerical results show that the PDF changes practically nothing for $n = 2$ and $n = 6$. In Fig. 3 we show the MSD (11) in function of time t; it presents anomalous diffusion processes with logarithmic oscillations. We see that the main trends in the MSD have power-law behaviors which indicate subdiffusive regimes.

We now consider that the deterministic drift $h(x, t)$ and multiplicative noise term $g(x, t)$ are separable in time and space, and they are given by

$$h(x, t) = T_1(t)\, D(x) \tag{16}$$

and

$$g(x, t) = T(t) D(x). \tag{17}$$

Then, Eq. (3) reduces to

$$\frac{\partial W(x,t)}{\partial t} = -T_1(t) \frac{\partial}{\partial x}\left[D(x) W(x,t)\right] + T^2(t) \frac{\partial}{\partial x}\left[D(x) \frac{\partial D(x) W(x,t)}{\partial x}\right] . \tag{18}$$

We note that the coefficients $h(x, t)$ and $g(x, t)$ given by $h(x, t) = g(x, t) = D(x)$ have been used for studying Brownian pumping in nonequilibrium transport processes [14]. By suitable transformations of variables we can show that Eq. (18) can be reduced to the constant-diffusion equation without the drift coefficient term. To do so, we take the following transformations:

$$\rho(x, t) = D(x) W(x, t) \ , \tag{19}$$

$$\frac{dt^*}{dt} = T^2(t) \tag{20}$$

and

$$x^* = \int \frac{\mathrm{d}x}{D(x)} - \int \mathrm{d}t T_1(t) + A, \tag{21}$$

where A is a constant, then Eq. (18) reduces to

$$\frac{\partial \rho(t^*, x^*)}{\partial t^*} = \frac{\partial^2 \rho(t^*, x^*)}{\partial x^{*2}} \ . \tag{22}$$

Eqs. (20) and (21) give the time and space scaling factors which connect Eq. (18) to the ordinary diffusion equation (22). Eq. (22) can be solved and the solution with a natural boundary condition is given by

$$\rho(t^*, x^*) = C \frac{\exp\left[-\frac{x^{*2}}{4t^*}\right]}{\sqrt{t^*}} \tag{23}$$

where C is a normalization factor. Eqs. (19) and (23) show that the time-dependent coefficients $T(t)$ and $T_1(t)$ do not change the Gaussian form, however the coefficient $D(x)$ can produce different forms for the distribution $W(x,t)$ [9]. We note that for $D(x)=\sqrt{D}$, $T(t)=1$ and $T_1(t)=0$ the Wiener process is recovered.

In order to investigate some details of the solution (23) we take $D(x)=\sqrt{D}\,|x|^{-\frac{\theta}{2}}$. From Eqs. (19) and (21), with $A=0$, yields

$$W(x,t)=\frac{(-x)^{\frac{\theta}{2}}}{\sqrt{4\pi D t^*(t)}}\exp\left[-\frac{\left((-x)^{\frac{2+\theta}{2}}+\frac{\sqrt{D}(2+\theta)}{2}H(t)\right)^2}{D(2+\theta)^2 t^*(t)}\right],\quad x<0\,,$$

$$W(x,t)=\frac{x^{\frac{\theta}{2}}}{\sqrt{4\pi D t^*(t)}}\exp\left[-\frac{\left(x^{\frac{2+\theta}{2}}-\frac{\sqrt{D}(2+\theta)}{2}H(t)\right)^2}{D(2+\theta)^2 t^*(t)}\right],\quad x>0\,, \tag{24}$$

where $H(t)=\int \mathrm{d}t T_1(t)$. Eq. (24) shows that the drift term produces an asymmetric PDF with respect to the coordinate x. For $T_1(t)=0$ the PDF (24) reduces to the solution (10) without the presence of the drift term, and the symmetric PDF is recovered. In this case, the drift term $T_1(t)$ gives the duration of this asymmetry. In Fig. 4 we show the asymmetric PDF Eq. (24) for $t=0.2$. The asymmetry of the PDF with $\theta=-0.1$ is more pronounced than the PDF with $\theta=-0.5$. From Eq. (24) we obtain

$$\left\langle x^2(t)\right\rangle=\frac{\Gamma\left(\frac{6+\theta}{2(2+\theta)}\right)}{\sqrt{\pi}}\left[D\left(2+\theta\right)^2 t^*(t)\right]^{\frac{2}{2+\theta}} e^{-\frac{H^2(t)}{t^*(t)}}\;{}_1F_1\left(\frac{6+\theta}{2\left(2+\theta\right)},\frac{1}{2},\frac{H^2(t)}{t^*(t)}\right), \tag{25}$$

where ${}_1F_1(a,b,z)$ is the Kummer confluent hypergeometric function [15]. For $H(t)=0$, without the drift term, we recover the result Eq. (11). Moreover, Eqs. (24) and (25) also present an interesting result; for $H^2(t)/t^*(t)$ proportional to a constant they give the similar results of Eqs. (10) and (11), without the drift term. In this case, the drift term only contributes an additional constant to the overall behavior of the system.

We should note that the solutions (19) and (23) can adequately work for $D(x)$ positive. For $D(x)$ containing negative values we should modify and take $\rho(x,t)=-D(x)W(x,t)$ for $D(x)$ negative. For example, let us consider $D(x)=x$. Then we take

$$\rho(x,t)=-xW(x,t),\; x<0 \tag{26}$$

and

$$\rho(x,t)=xW(x,t),\; x>0\;. \tag{27}$$

From Eq. (21) we obtain

$$x^{*}=\ln|x|-H(t)-\ln|x_{0}| \tag{28}$$

and

$$W(x,t)=\frac{\exp\left[-\frac{(\ln|x|-H(t)-\ln|x_{0}|)^{2}}{4t^{*}(t)}\right]}{4\sqrt{\pi t^{*}(t)}\,|x|}\ . \tag{29}$$

This is the log-normal distribution. The distribution (29) is the same as the one given in Ref. [16] for $t^{*}(t)=t$, which has been obtained from the method of characteristic. It is worth mentioning that the coefficients $h(x,t)$ and $g(x,t)$ given by $h(x,t)\sim x$ and $g(x,t)\sim x$ might be used to investigate the barrier crossing problem in heavy-ion fusion reaction [17], and they can also be used as a limiting case of a Langevin equation for describing the tumor cell growth system [18].

Conclusion. When a multiplicative noise term is introduced into the simple Langevin equation (2), even separable in time and space, the system can exhibit complex behaviors and a rich variety of processes. We have analytically presented a class of these processes. We hope that they can be used to mimic a wide class of natural systems.

Acknowledgment

The author acknowledges partial financial support from the Conselho Nacional de Desenvolvimento Científico e Tecnológico (CNPq), Brazilian agency.

[1] Metzler R. and Klafter J., *Phys. Rep.,* **339,** (2000) 1; Metzler R. and Klafter J., *J. Phys. A: Math. Gen.*, **37** (2004) R161.

[2] U. M. B. Marconi, A. Puglisi, L. Rondoni, and A. Vulpiani, Phys. Rep. 461, 111 2008; U. Balucani, M. H. Lee, and V. Tognetti, Phys. Rep. 373, 409 (2003).

[3] H. Risken, *The Fokker-Planck Equation* (Springer-Verlag, Berlin, 1996).

[4] C.W. Gardiner, *Handbook of Stochastic Methods* (Springer-Verlag, Berlin, 1997).

[5] K. S. Fa and K. G. Wang, Phys. Rev. E **81,** 011126 (2010).

[6] L. F. Richardson, Proc. R. Soc. London, Ser. A **110,** 709 (1926).

[7] G. K. Batchelor, Proc. Cambridge Philos. Soc. **48,** 345 (1952).

[8] H. G. E. Hentschel and I. Procaccia, Phys. Rev. A **29**, 1461 (1984).

[9] K. S. Fa, Phys. Rev. E **72,** 020101 (2005).

[10] F. Iglói, L. Turban, and H. Rieger, Phys. Rev. E 59, 1465 (1999).

[11] M.A. Bab, E.V. Albano, J. Phys. A: Math. Theor. **41**, 045001 (2008).

[12] N. Vandewalle, Ph. Boveroux, A. Minguet, M. Ausloos, Physica A **255**, 201 (1998).

[13] D. Sornette, A. Johansen, A. Arneodo, J.F. Muzy, H. Saleur, Phys. Rev. Lett. **76**, 251 (1996).

[14] A. Gomez-Marin and J. M. Sancho, Phys. Rev. E **77**, 031108 (2008).

[15] S. Wolfram, *The Mathematica*, 3rd Ed. (Cambridge Univ. Press, 1996)

[16] K. S. Fa, Chem. Phys. **287**, 1 (2003).

[17] J. W. Mao and C. W. Shen, Phys. Rev. E **83**, 041108 (2011).

[18] B. Q. Ai, X. J. Wang, G. T. Liu and L. G. Liu, Phys. Rev. E **67**, 022903 (2003); C. J. Wang, Q. Wei and D. C. Mei, Modern Physics Letters B **21**, 789 (2007).

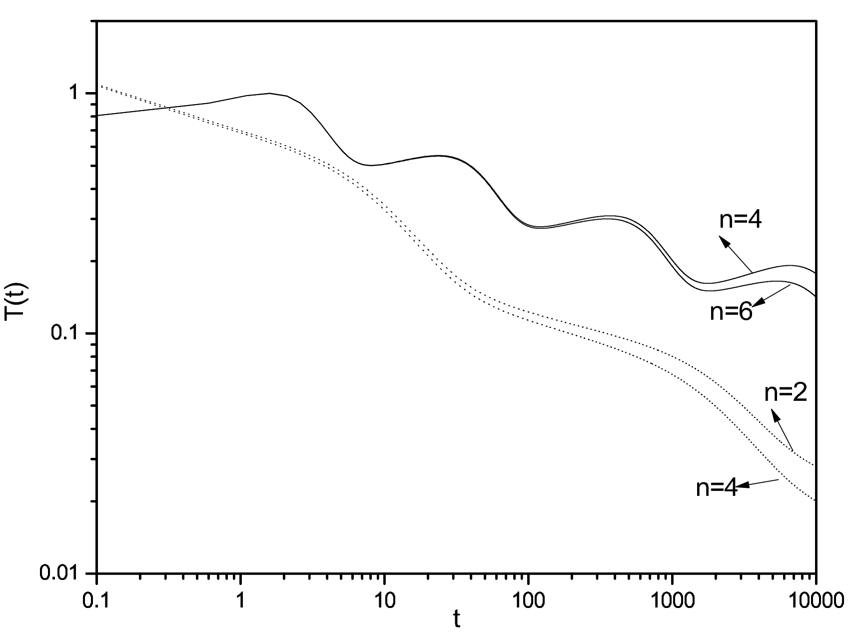
FIGURE CAPTIONS

FIG. 1 - Plots of the function $T(t)$, Eq. (14). The dashed lines correspond to $\alpha = 0.5$, whereas the solid lines correspond to $\alpha = 1$.

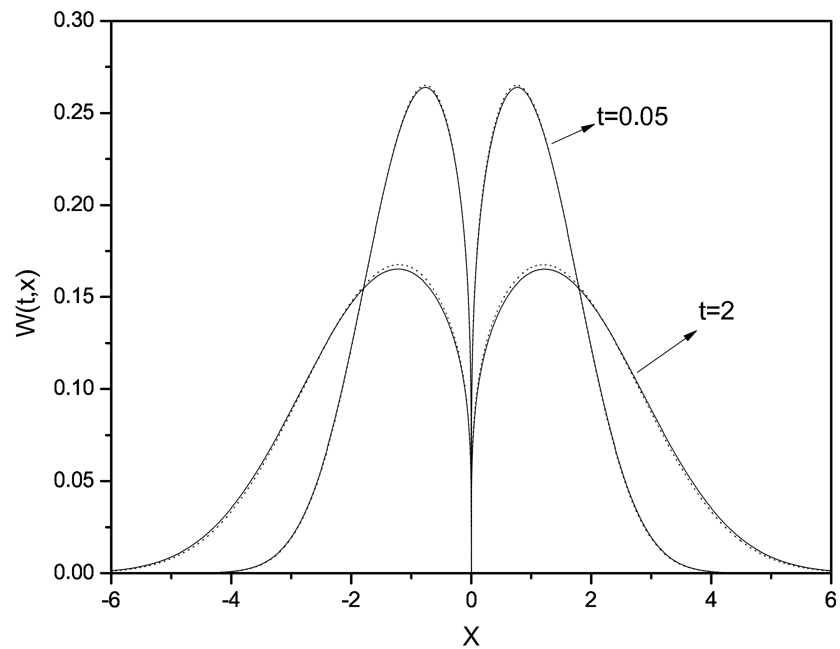
FIG. 2 - Plots of the PDF (10) for $\lambda_i = a^i$, $c_i = (a/b)^i$, $a = 1/15$, $b = 0.3$, $D = 1$, $\theta = 0.5$ and $\alpha = 1$. The solid lines correspond to $n = 2$, whereas the dotted lines correspond to $n = 6$.

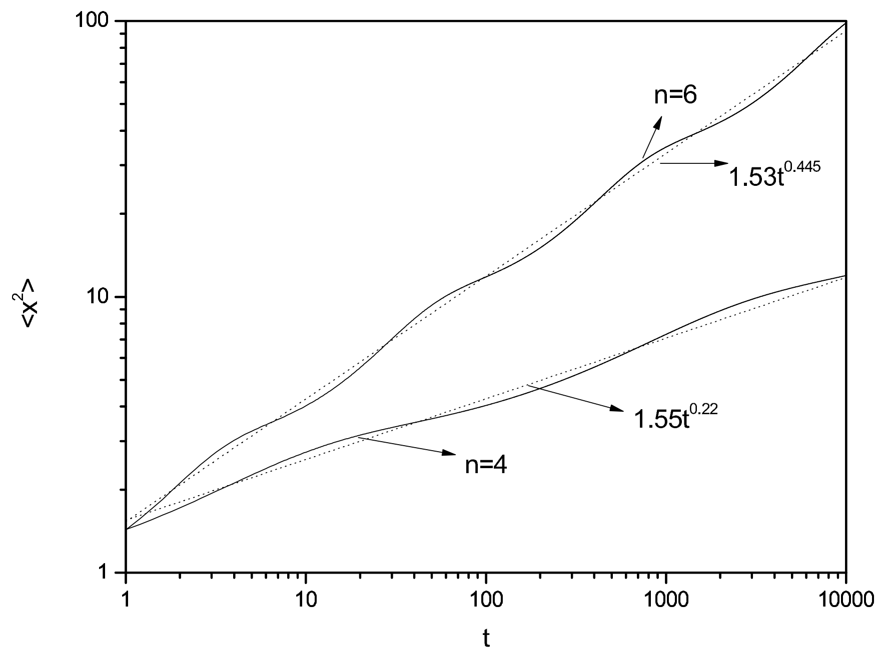
FIG. 3 - Plots of the MSD (11) for $\lambda_i = a^i$, $c_i = (a/b)^i$, $a = 1/15$, $b = 0.3$, $D = 1$ and $\theta = 0.5$. The solid line with $n = 4$ corresponds to $\alpha = 0.5$, whereas the solid line with $n = 6$ corresponds to $\alpha = 1$. The dashed lines correspond to the power-law functions.

FIG. 4 - Plots of the PDF (24) for $D = 1$, $t^*(t) = t$ and $H(t) = t$. The dotted line corresponds to $\theta = -0.5$, whereas the solid line corresponds to $\theta = -0.1$.

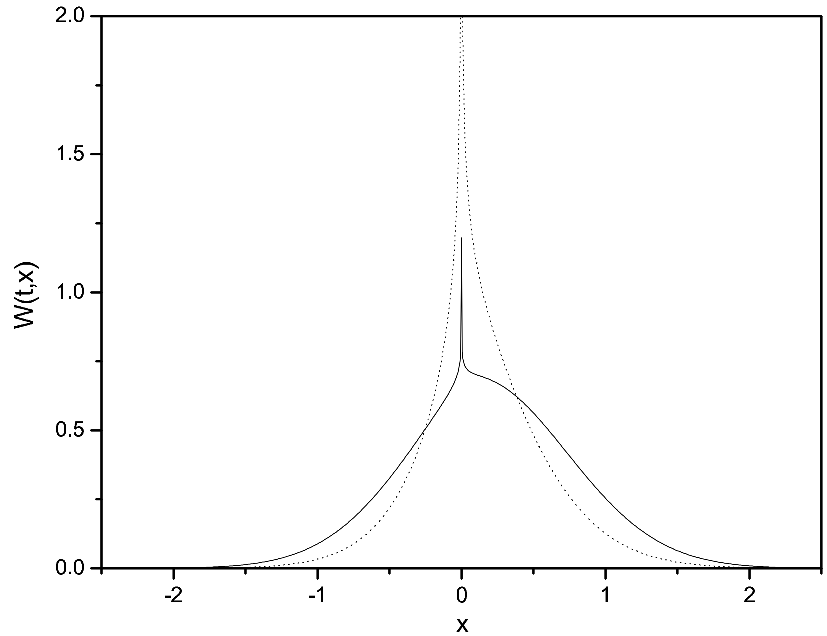


T(t)
t
1
0.1
0.01
0.1
1
10
100
1000
10000
n=4
n=6
n=2
n=4


0.30
0.25
0.20
0.15
0.10
0.05
0.00
W(t,x)
-6
-4
-2
0
2
4
6
X
t=0.05
t=2



100
10
1
<x2>
1
10
100
1000
10000
t
n=6
1.53t0.445
n=4
1.55t0.22


W(t,x)
2.0
1.5
1.0
0.5
0.0
-2
-1
0
1
2
x